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Numerical algorithm for parabolic problems with non-classical conditions☆

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ABSTRACT

The parabolic problems with non-classical conditions are discussed in a reproducing kernel space in this paper. A reproducing kernel space is constructed, in which the non-classical conditions of the parabolic problems are satisfied. Based on the reproducing kernel space, a new technique for solving the non-classical parabolic problems is presented. Some examples are displayed to demonstrate the validity and applicability of the proposed method.

1. Introduction

Non-classical boundary value problems with nonlocal boundary conditions arise naturally in various engineering models and physical phenomena. Such problems gained much attention in recent years, not only in engineering but also in the mathematics community. They have many important applications in chemical diffusion, thermoelasticity, heat conduction processes, population dynamics, nuclear reactor dynamics, inverse problems, control theory, medical science, biochemistry and certain biological processes, and so forth (see [2–6] and the references therein).

Parabolic initial-boundary value problems in one dimension that involve nonstandard boundary conditions are studied by several authors. A number of articles addressed the question of existence and uniqueness of the solutions and theoretical discussion of the one-dimensional parabolic equation with nonstandard boundary conditions [7–13]. There has been growing interest in developing computational techniques for the numerical solution of the one-dimensional parabolic partial differential equations with nonlocal boundary specifications. The familiar finite difference schemes, finite element techniques, spectral procedures, Adomian decomposition method, Chebyshev spectral collocation method, boundary element methods, integro-differential approach and so on (see [14–20] and the references therein) have been used for the one-dimensional parabolic equation with nonlocal boundary conditions.

Ang [21] proposed an alternative method of solution one not based on finite difference approximations for solving the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < 1, 0 < t \leq T) \tag{1.1}$$

with initial condition

$$u(x, 0) = f(x) \tag{1.2}$$

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and nonlocal boundary conditions.

$$u(0,t) = \int_0^1 h_0(x)u(x,t)\mathrm{d}x + g_0(t), \quad (0 < t \leq T) \tag{1.3}$$

$$u(1,t) = \int_0^1 h_1(x)u(x,t)\mathrm{d}x + g_1(t), \quad (0 < t \leq T) \tag{1.4}$$

where f, g_0, g_1, h_0 and h_1 are prescribed functions. By taking the Laplace transformations of (1.1) and (1.3), (1.4) with respect to the time coordinate, the problem is solved. In the Laplace transform domain, the problem can be reduced to a boundary value problem that is governed by a second-order inhomogeneous ordinary differential equation and can be solved explicitly.

The author of [22] studied the problem

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < 1, 0 < t \leq T) \tag{1.5}$$

with the initial condition (1.2) and nonlocal conditions (1.3), (1.4). He proposed the θ-method for the problem. The systems of linear equations generated by this method have a coefficient matrix that is tridiagonal except its first and last rows. New algorithms for solving this kind of linear systems are proposed. One of them is a straightforward adaption of the well-known Thomas algorithm for solving tridiagonal systems of equations. The other two are based on the inverse matrix modification formulae. A numerical example is given to compare the efficiency.

The authors of [23] have developed a numerical method for the solution of (1.5) with initial condition (1.2) and nonlocal boundary specifications

$$u(0,t) = \int_0^1 \phi(x)u(x,t)\mathrm{d}x, \quad (0 < t \leq T) \tag{1.6}$$

$$u(1,t) = \int_0^1 \psi(x)u(x,t)\mathrm{d}x, \quad (0 < t \leq T) \tag{1.7}$$

where f, q, ϕ and ψ are prescribed functions. They have assumed the data to be sufficiently smooth and to satisfy the compatibility condition. They have chosen the Crank–Nicolson method, which is unconditionally stable, has a very decent second-order accuracy and describes the diffusion process faithfully, as the frame work of their scheme. Simpson's numerical integration formula, which has a truncation error of higher order than the trapezoidal rule, is also employed.

The author of [24] developed and analyzed three different finite difference methods for solving the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \quad (0 < x < 1, 0 < t \leq T) \tag{1.8}$$

with initial condition (1.2) and nonlocal boundary conditions

$$u(0,t) = \int_0^1 h_0(x)u(x,t)\mathrm{d}x + g_0(t), \quad (0 < t \leq T) \tag{1.9}$$

$$u(1,t) = \int_0^1 h_1(x)u(x,t)\mathrm{d}x + g_1(t), \quad (0 < t \leq T) \tag{1.10}$$

where f, g_0, g_1, q, h_0 and h_1 are prescribed functions. The schemes which have been developed in [24] for the above heat equation in one space dimension with boundary conditions containing integrals over the interior of the interval are based on the forward Euler scheme, the backward Euler technique and the Crank–Nicolson method. The integrals in the boundary conditions for $x = 0, 1$ are approximated by the trapezoidal rule. He has chosen this approximation since it is simple and of the same second-order accuracy in space as the methods used for the interior part of the problem. Wang and Lin's treatment for the nonlocal specifications (1.9) and (1.10), produces a linear system which is not tridiagonal, since the first and last rows are not in the required form. Taking advantage of this special form the system can easily be transformed into an upper triangular system. This method costs the same as the usual Gaussian rule with pivoting [23]. They have demonstrated two numerical experiments. The author [25] discuss (1.8) with initial condition (1.2) and boundary value condition (1.6), (1.7). The method of lines semi-discretization approach will be used to transform the model partial differential equation into a system of first-order linear ordinary differential equations. The spatial derivative in the PDE is approximated by a finite-difference approximation. The solution of the resulting system of first-order ordinary differential equations satisfies a recurrence relation which involves a matrix exponential function. Numerical techniques are developed by approximating the exponential matrix function in this recurrence relation.

An excellent overview of recent developments is found in [26,1]. A more extensive list of references as well as a survey on progress made on the nonlocal boundary value problems may be found in [27].

This work is aimed at producing a very efficient technique for solving the following non-classical parabolic problems based on the reproducing kernel space

$$u_t - u_{xx} = g(x,t), \quad (0 < x < 1, 0 < t < T) \tag{1.11}$$

subject to the initial condition

$$u(x,0) = f(x), \quad (0 < x < 1) \tag{1.12}$$

and the nonlocal boundary conditions

$$\begin{aligned} \lambda_0 u(0,t) + \gamma_0 u_x(0,t) &= \int_0^1 p_0(x)u(x,t)\mathrm{d}x + q_0(t), \quad (0 < t < T) \\ \lambda_1 u(1,t) + \gamma_1 u_x(1,t) &= \int_0^1 p_1(x)u(x,t)\mathrm{d}x + q_1(t), \quad (0 < t < T) \end{aligned} \tag{1.13}$$

where g, f, p_i, q_i $(i = 0, 1)$ are sufficiently smooth prescribed functions, and λ_0, γ_0, λ_1, γ_1 are given constants.

A numerical method based on Chebyshev polynomials and local interpolating functions [14] is proposed for solving Eqs. (1.11)–(1.13). In [1], the author presented two-level Crandall's implicit scheme, the three-level Dufort–Frankel explicit method, BTCS implicit method and FTCS method for solving Eqs. (1.11)–(1.13). In this paper, we solve Eqs. (1.11)–(1.13) based on the reproducing kernel space. The advantage of the approach lies in the fact that the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives.

The outline of this paper is as follows. Several reproducing kernel spaces are defined in Section 2. Section 3 introduces the main results in the reproducing kernel space. The algorithm of Eq. (1.11) with (1.12), (1.13) is given, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. In Section 4, several numerical experiments are provided to show the validity and applicability of the method developed in this article. Section 5 is devoted to a brief conclusion.

2. Preliminaries

In this section, several reproducing kernel spaces are defined.

2.1. The reproducing kernel space $W_0[0,1]$, $W_1[0,T]$, $W_2[0,1]$

Definition 2.1. $W_0[0,1] = \{u(x) \mid u, u', u''$ are one-variable absolutely continuous real-valued functions in $[0,1]$, $u''' \in L^2[0,1]$, $\lambda_0 u(0) + \gamma_0 u'(0) - \int_0^1 p_0(x)u(x)\mathrm{d}x = 0$, $\lambda_1 u(1) + \gamma_1 u'(1) - \int_0^1 p_1(x)u(x)\mathrm{d}x = 0\}$, where λ_0, γ_0, λ_1, γ_1, $p_0(x)$, $p_1(x)$ are given in (1.13).

$W_0[0,1]$ is a Hilbert space, the inner product and the norm in $W_0[0,1]$ are given by

$$\langle u, v\rangle_{W_0} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u'''v'''\mathrm{d}x, \qquad \|u\|_{W_0} = \sqrt{\langle u, u\rangle_{W_0}}, \tag{2.1}$$

respectively, where $u, v \in W_0[0,1]$.

The space $W_0[0,1]$ is a reproducing kernel space. That is, there exist a reproducing kernel function $R_y(x) \in W_0[0,1]$, $x \in [0,1]$, for each fixed $y \in [0,1]$ and any $u(x) \in W_0[0,1]$, such that

$$\langle u(x), R_y(x)\rangle_{W_0} = u(y),$$

and $R_y(x)$ can be obtained in Appendix.

Definition 2.2. $W_1[0,T] = \{u(t) \mid u, u'$ are one-variable absolutely continuous real-valued functions in $[0,T]$, $u'' \in L^2[0,T]$, $u(0) = 0\}$.

$W_1[0,T]$ is a Hilbert space, the inner product and the norm in $W_1[0,T]$ are given by

$$\langle u, v\rangle_{W_1} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^T u''v''\mathrm{d}t, \qquad \|u\|_{W_1} = \sqrt{\langle u, u\rangle_{W_1}}, \tag{2.2}$$

respectively, where $u, v \in W_1[0,T]$. The space $W_1[0,T]$ is a reproducing kernel space. Its reproducing kernel function $r_s(t)$ is given by

$$r_s(t) = \begin{cases} -\dfrac{1}{6}t(t^2 - 3s(2+t)), & t \le s, \\ -\dfrac{s^3}{6} + \dfrac{1}{2}s(2+s)t, & t > s. \end{cases} \tag{2.3}$$

Definition 2.3. $W_2[0,1] = \{u(x) \mid u$ is one-variable absolutely continuous real-valued function in $[0,1], u' \in L^2[0,1]\}$.

$W_2[0,1]$ is a Hilbert space, the inner product and the norm in $W_2[0,1]$ are given by

$$\langle u, v\rangle_{W_2} = u(0)v(0) + \int_0^1 u'v'\mathrm{d}x, \qquad \|u\|_{W_2} = \sqrt{\langle u, u\rangle_{W_2}}, \tag{2.4}$$

respectively, where $u, v \in W_2[0,1]$. The space $W_2[0,1]$ is a reproducing kernel space. Its reproducing kernel function $Q_y(x)$ is given by

$$Q_y(x) = \begin{cases} 1+x, & x \le y, \\ 1+y, & x > y. \end{cases} \tag{2.5}$$

2.2. The reproducing kernel space $W(\Omega)$, $\widetilde{W}(\Omega)$, $\Omega = [0,1]\times[0,T]$

Definition 2.4. $W(\Omega) = \{u(x,t) \mid \partial_x^2\partial_t u(x,t)$ is complete continuous in Ω, $\partial_x^3\partial_t^2 u(x,t) \in L^2(\Omega), u(x,0) = 0, \lambda_0 u(0,t) + \gamma_0 u_x(0,t) - \int_0^1 p_0(x)u(x,t)\mathrm{d}x = 0, \lambda_1 u(1,t) + \gamma_1 u_x(1,t) - \int_0^1 p_1(x)u(x,t)\mathrm{d}x = 0\}$, where $\lambda_0, \gamma_0, \lambda_1, \gamma_1, p_0(x), p_1(x)$ are given in (1.13).

$W(\Omega)$ is a Hilbert space, for $u, v \in W(\Omega)$, the inner product is defined by

$$\begin{aligned}\langle u(x,t), v(x,t)\rangle_W = &\sum_{j=0}^{1}\langle \partial_t^j u(x,0), \partial_t^j v(x,0)\rangle_{W_0} + \sum_{i=0}^{2}\int_0^T [\partial_t^2\partial_x^i u(0,t)\partial_t^2\partial_x^i v(0,t)]\mathrm{d}t \\ &+ \iint_\Omega [\partial_x^3\partial_t^2 u(x,t)\partial_x^3\partial_t^2 v(x,t)]\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.6}$$

and the norm is

$$\|u\|_W = \sqrt{\langle u, u\rangle_W}.$$

Theorem 2.1. *$W(\Omega)$ is a reproducing kernel space and its reproducing kernel function is*

$$K_{(y,s)}(x,t) = R_y(x)r_s(t), \tag{2.7}$$

for any $u(x,t) \in W(\Omega)$, such that

$$u(y,s) = \langle u(x,t), K_{(y,s)}(x,t)\rangle_W \tag{2.8}$$

and

$$K_{(y,s)}(x,t) = K_{(x,t)}(y,s), \tag{2.9}$$

where $R_y(x)$, $r_s(t)$ are the reproducing kernel functions of $W_0[0,1]$ and $W_1[0,1]$, respectively.

Proof. Note that

$$\begin{aligned}\langle u(x,t), R_y(x)r_s(t)\rangle_W &= \sum_{j=0}^{1}\langle \partial_t^j u(x,0), R_y(x)\partial_t^j r_s(0)\rangle_{W_0} + \sum_{i=0}^{2}\int_0^T [\partial_t^2\partial_x^i u(0,t)\partial_t^2 r_s(t)\partial_x^i R_y(0)]\mathrm{d}t \\ &\quad + \iint_\Omega [\partial_x^3\partial_t^2 u(x,t)\partial_x^3 R_y(x)\partial_t^2 r_s(t)]\mathrm{d}x\mathrm{d}t \\ &= \sum_{j=0}^{1}\partial_t^j u(y,0)\partial_t^j r_s(0) + \int_0^T \partial_t^2 r_s(t)\partial_t^2\left[\int_0^1 \partial_x^3 u(x,t)\partial_x^3 R_y(x)\mathrm{d}x + \sum_{i=0}^{2}\partial_x^i u(0,t)\partial_x^i R_y(0)\right]\mathrm{d}t \\ &= \sum_{j=0}^{1}\partial_t^j u(y,0)\partial_t^j r_s(0) + \int_0^T \partial_t^2 r_s(t)\partial_t^2\langle u(x,t), R_y(x)\rangle_{W_0}\mathrm{d}t \\ &= \sum_{j=0}^{1}\partial_t^j u(y,0)\partial_t^j r_s(0) + \int_0^T \partial_t^2 r_s(t)\partial_t^2 u(y,t)\mathrm{d}t \\ &= \langle u(y,t), r_s(t)\rangle_{W_1} = u(y,s)\end{aligned}$$

thus $u(y,s) = \langle u(x,t), K_{(y,s)}(x,t)\rangle_W$, and

$$K_{(y,s)}(x,t) = \langle K_{(y,s)}(\xi,\eta), K_{(x,t)}(\xi,\eta)\rangle_W = \langle K_{(x,t)}(\xi,\eta), K_{(y,s)}(\xi,\eta)\rangle_W = K_{(x,t)}(y,s).$$

So the proof is completed. □

Now the definition of $\widetilde{W}(\Omega)$ is given.

Definition 2.5. $\widetilde{W}(\Omega) = \{u(x,t) \mid u(x,t)$ is complete continuous in Ω, $\partial^2_{xt}u(x,t) \in L^2(\Omega)\}$.

$\widetilde{W}(\Omega)$ is a Hilbert space, for $u, v \in \widetilde{W}(\Omega)$, the inner product is given by

$$\langle u(x,t), v(x,t)\rangle_{\widetilde{W}} = \langle u(x,0), v(x,0)\rangle_{W_2} + \int_0^T [\partial_t u(0,t)\partial_t v(0,t)]\mathrm{d}t + \iint_\Omega [\partial^2_{xt}u(x,t)\partial^2_{xt}v(x,t)]\mathrm{d}x\mathrm{d}t \tag{2.10}$$

and the norm is $\|u\|_{\widetilde{W}} = \sqrt{\langle u,u\rangle_{\widetilde{W}}}$. $\widetilde{W}(\Omega)$ is a reproducing kernel space and its reproducing kernel function is $G_{(y,s)}(x,t) = Q_y(x)q_s(t)$, where $Q_y(x)$ given by (2.5), and

$$q_s(t) = \begin{cases} 1+t, & t \le s, \\ 1+s, & t > s. \end{cases}$$

3. The main results

3.1. Homogenization of the initial condition and nonlocal boundary conditions

In order to Eq. (1.11) subject to (1.12) and (1.13) can be involved in the reproducing kernel space $W(\Omega)$, we only need homogenize the initial condition and nonlocal boundary conditions.

Letting $\tilde{u}(x,t) = u(x,t) - E(x,t) - f(x) + E(x,0)$, then Eq. (1.11) with (1.12), (1.13) can be converted into the following form

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = h(x,t), \quad (0 < x < 1, 0 < t < T) \\ \tilde{u}(x,0) = 0, \quad (0 < x < 1) \\ \lambda_0\tilde{u}(0,t) + \gamma_0\tilde{u}_x(0,t) - \displaystyle\int_0^1 p_0(x)\tilde{u}(x,t)\mathrm{d}x = 0, \quad (0 < t < T) \\ \lambda_1\tilde{u}(1,t) + \gamma_1\tilde{u}_x(1,t) - \displaystyle\int_0^1 p_1(x)\tilde{u}(x,t)\mathrm{d}x = 0, \quad (0 < t < T) \end{cases} \tag{3.1}$$

where $h(x,t) = g(x,t) - \partial_t(E(x,t) + f(x) - E(x,0)) + \partial^2_{xx}(E(x,t) + f(x) - E(x,0))$, $E(x,t) = q_0(t)(ax^2 + bx + c) + q_1(t)(Ax^2 + Bx + d)$, and a, b, c, A, B, d are constants. In (3.1), without loss of generality, denote $\tilde{u}(x,t)$ by $u(x,t)$ in the following discussion.

3.2. Implementation method to solve Eq. (3.1) in $W(\Omega)$

Define linear operator $\mathbb{L} : W(\Omega) \to \widetilde{W}(\Omega)$. For any $u(x,t) \in W(\Omega)$,

$$\mathbb{L}u(x,t) = u_t - u_{xx},$$

then Eq. (3.1) can be converted into the form as follows

$$\begin{cases} \mathbb{L}u(x,t) = h(x,t), \quad (0 < x < 1, 0 < t < T) \\ u(x,0) = 0, \quad (0 < x < 1) \\ \lambda_0 u(0,t) + \gamma_0 u_x(0,t) - \displaystyle\int_0^1 p_0(x)u(x,t)\mathrm{d}x = 0, \quad (0 < t < T) \\ \lambda_1 u(1,t) + \gamma_1 u_x(1,t) - \displaystyle\int_0^1 p_1(x)u(x,t)\mathrm{d}x = 0, \quad (0 < t < T) \end{cases} \tag{3.2}$$

where $u(x,t) \in W(\Omega)$ and $h(x,t) \in \widetilde{W}(\Omega)$ as $u = u(x,t) \in W(\Omega)$.

In the contexts, C_i $(i = 1, 2, 3, 4, 5, 6)$ are constants.

Lemma 3.1. *$\mathbb{L} : W(\Omega) \to \widetilde{W}(\Omega)$ is a bounded linear operator.*

Proof. In view of (2.10),

$$\begin{aligned} \|\mathbb{L}u(x,t)\|^2_{\widetilde{W}} &= \langle \mathbb{L}u(x,t), \mathbb{L}u(x,t)\rangle_{\widetilde{W}} \\ &= \int_0^T [\partial_t \mathbb{L}u(0,t)]^2\mathrm{d}t + \langle \mathbb{L}u(x,0), \mathbb{L}u(x,0)\rangle_{W_2} + \iint_\Omega [\partial^2_{xt}\mathbb{L}u(x,t)]^2\mathrm{d}x\mathrm{d}t \\ &= \int_0^T [\partial_t \mathbb{L}u(0,t)]^2\mathrm{d}t + [\mathbb{L}u(0,0)]^2 + \int_0^1 [\partial_x \mathbb{L}u(x,0)]^2\mathrm{d}x + \iint_\Omega [\partial^2_{xt}\mathbb{L}u(x,t)]^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

From the continuous of $K_{(y,s)}(x,t)$ and (2.9), we have

$$\|\mathbb{L}K_{(x,t)}(\circ,\star)\|_W = \|\partial_t K_{(x,t)}(\circ,\star) - \partial^2_{xx}K_{(x,t)}(\circ,\star)\|_W \le C_1.$$

Note that

$$u(x,t) = \langle u(\circ,\star), K_{(x,t)}(\circ,\star)\rangle_W, \qquad \mathbb{L}u(x,t) = \langle u(\circ,\star), \mathbb{L}K_{(x,t)}(\circ,\star)\rangle_W,$$

thus

$$|\mathbb{L}u(x,t)| \le \|u\|_W \|\mathbb{L}K_{(x,t)}(\circ,\star)\|_W \le C_1\|u\|_W,$$
$$[\mathbb{L}u(0,0)]^2 \le C_1^2\|u\|_W^2.$$

Similarly, we have

$$\partial_t\mathbb{L}u(x,t) = \langle u(\circ,\star), \partial_t\mathbb{L}K_{(x,t)}(\circ,\star)\rangle_W,$$
$$\partial^2_{xx}\mathbb{L}u(x,t) = \langle u(\circ,\star), \partial^2_{xx}\mathbb{L}K_{(x,t)}(\circ,\star)\rangle_W,$$

then

$$|\partial_t\mathbb{L}u(x,t)| \le C_2\|u\|_W, \qquad |\partial^2_{xx}\mathbb{L}u(x,t)| \le C_3\|u\|_W,$$

furthermore

$$[\partial_t\mathbb{L}u(0,t)]^2 \le C_2^2\|u\|_W^2, \qquad [\partial^2_{xx}\mathbb{L}u(x,t)]^2 \le C_3^2\|u\|_W^2.$$

Combining the above argument yields

$$\|\mathbb{L}u(x,t)\|^2_{\widetilde{W}} \le (C_1^2 + C_2^2 + C_3^2)\|u\|_W^2 \le C_4\|u\|_W^2. \quad \square$$

Let a countable dense subset $\{M_i\}_{i=1}^{\infty} = \{(x_i,t_i)\}_{i=1}^{\infty} \subset \Omega$ and $M = (x,t)$. Taking

$$\varphi_i(M) = G_{M_i}(M), \qquad \psi_i(M) = \mathbb{L}^*\varphi_i(M),$$

where $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$.

Lemma 3.2. *Assume $\{M_i\}_{i=1}^{\infty}$ be dense in Ω, then $\{\psi_i(M)\}_{i=1}^{\infty}$ is the complete system in $W(\Omega)$.*

Proof. Since

$$\begin{aligned}\psi_i(M) &= (\mathbb{L}^*\varphi_i)(M) = \langle(\mathbb{L}^*\varphi_i)(M'), K_M(M')\rangle_W \\ &= \langle\varphi_i(M'), \mathbb{L}_{M'}K_M(M')\rangle_{\widetilde{W}} = \mathbb{L}_{M'}K_M(M')\,|_{M'=M_i}.\end{aligned}$$

Clearly, $\psi_i(M) \in W(\Omega)$. For any $u(M) \in W(\Omega)$, let

$$\langle u(M), \psi_i(M)\rangle_W = 0 \quad (i = 1,2,\ldots)$$

which means that,

$$\langle u(M), \mathbb{L}^*\varphi_i(M)\rangle_W = \langle\mathbb{L}u(\cdot,\star), \varphi_i(\cdot,\star)\rangle_{\widetilde{W}} = (\mathbb{L}u)(M_i) = 0.$$

Note that $\{M_i\}_{i=1}^{\infty}$ is dense in Ω, hence $\mathbb{L}u(M) = 0$. It follows that $u(M) \equiv 0$ from the existence of $\mathbb{L}^{-1}$.

So the proof is complete. $\square$

Denote the orthonormal system $\{\overline{\psi}_i(M)\}_{i=1}^{\infty}$ in $W(\Omega)$ which derives from Gram–Schmidt orthogonalization process of $\{\psi_i(M)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(M) = \sum_{k=1}^{i}\beta_{ik}\psi_k(M),$$

where β_{ik} are orthogonal coefficients.

The first main result of this paper is the following theorem, which provides the exact expression of the solution of Eq. (3.2) in $W(\Omega)$.

Theorem 3.1. *Suppose $\{M_i\}_{i=1}^{\infty}$ be dense in Ω, if $u(M) \in W(\Omega)$ is the solution of (3.2), then*

$$u(M) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[h(M_k)]\overline{\psi}_i(M), \tag{3.3}$$

and $u(M) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[h(M_k)]\overline{\psi}_i(M)$ is a convergent series in the sense of $\|\cdot\|_W$.

Proof. since $\{\overline{\psi}_i(M)\}_{i=1}^{\infty}$ is the complete system in $W(\Omega)$, then

$$\begin{aligned}
u(M) &= \sum_{i=1}^{\infty}\langle u(M), \overline{\psi}_i(M)\rangle_W \overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(M), \psi_k(M)\rangle_W \overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(M), \mathbb{L}^*\varphi_k(M)\rangle_W \overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \mathbb{L}u(M), \varphi_k(M)\rangle_{\widetilde{W}} \overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \mathbb{L}u(M), G_{M_k}(M)\rangle_{\widetilde{W}} \overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\mathbb{L}u(M_k)\overline{\psi}_i(M)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[h(M_k)]\overline{\psi}_i(M).
\end{aligned}$$

Note that $W(\Omega)$ is a Hilbert space, $\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}h(M_k)\overline{\psi}_i(M)$ is Fourier series in $W(\Omega)$, thus it is a convergent series in the sense of $\|\cdot\|_W$. So the proof is complete. □

Now we construct the approximate solution of Eq. (3.2).

We divide the domain $\Omega = [0, 1] \times [0, T]$ into a $p \times q$ mesh with the spatial step size $\Delta x = 1/p$ in x direction and the time step size $\Delta t = T/q$, respectively, in which p and q are integers. Grid points (x_l, t_m) are defined by

$$x_l = l \times \Delta x, (l = 0, 1, 2 \ldots p), \qquad t_m = m \times \Delta t, \quad (m = 0, 1, 2 \ldots q). \tag{3.4}$$

By truncating the series given in (3.3), we can obtain the approximate solution of Eq. (3.2) as follows

$$u_n(M) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}[h(M_k)]\overline{\psi}_i(M) \tag{3.5}$$

where $n = p \times q$, $M_k = (x_k, t_k)$ is given by (3.4) and β_{ij} are orthogonal coefficients given by

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \qquad \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \qquad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}^2\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}},$$

in which $c_{ik} = \langle\psi_i, \overline{\psi}_k\rangle_W$.

Denote the error between the approximate $u_n(M)$ and the exact solution $u(M)$ by

$$\varepsilon_n(M) = u(M) - u_n(M). \tag{3.6}$$

Theorem 3.2. *Assume that $u(M)$ is the solution of Eq. (3.2), $u_n(M)$ is the approximate solution of Eq. (3.2), where $u(M)$ and $u_n(M)$ are given by (3.3), (3.5), respectively, then $\|u(M) - u_n(M)\|_W \to 0$, $(n \to \infty)$ and $\|u(M) - u_n(M)\|_C \to 0$, $(n \to \infty)$.*

Proof. From (3.3) and (3.5), $W(\Omega)$ is a Hilbert space, it is easy to see that

$$\|u(M) - u_n(M)\|_W \to 0, (n \to \infty).$$

Since

$$\begin{aligned}
|u(M) - u_n(M)| &= |\langle u(M') - u_n(M'), K_M(M')\rangle_W|\\
&\le \|u(M) - u_n(M)\|_W \|K_M(M')\|_W\\
&\le C_5\|u - u_n\|_W \to 0, (n \to \infty),
\end{aligned}$$

thus

$$\|u(M) - u_n(M)\|_C \to 0, (n \to \infty). \quad \square$$

Theorem 3.3. *Assume that the conditions of Theorem 3.2 hold, then the derivatives of $u_n(M)$ converge to the derivatives of $u(M)$ uniformly, respectively.*

Proof. From Theorem 3.2, $u_n(M)$ converge to $u(M)$ uniformly, in view of

$$\begin{aligned}|\partial^2_{xx}u(M) - \partial^2_{xx}u_n(M)| &= |\partial^2_{xx}(u(M) - u_n(M))|\\ &= |\partial^2_{xx}\langle u(M') - u_n(M'), K_M(M')\rangle_W|\\ &= |\langle u(M) - u_n(M), \partial^2_{xx}K_M(M')\rangle_W|\\ &\le \|u(M') - u_n(M')\|_W\|\partial^2_{xx}K_M(M')\|_W\\ &\le C_6\|u - u_n\|_W \to 0, (n \to \infty),\end{aligned}$$

we have

$$\|\partial^2_{xx}u(M) - \partial^2_{xx}u_n(M)\|_C \to 0, (n \to \infty),$$

namely, $\partial^2_{xx}u_n(M)$ converge to $\partial^2_{xx}u(M)$ uniformly. Similarly, others can be proved. □

Theorem 3.4. *Assume that the conditions of Theorem 3.2 hold, then the error $\varepsilon_n(M)$ given by (3.6) is monotone decreasing in the sense of $\|\cdot\|_W$, and $\|\varepsilon_n\|_W \to 0$, $(n \to \infty)$.*

Proof. By (3.6), we have

$$\|\varepsilon_n\|^2_W = \|u - u_n\|^2_W = \sum_{i=n+1}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}h(M_k)\right)^2,$$

$$\|\varepsilon_{n+1}\|^2_W = \|u - u_n\|^2_W = \sum_{i=n+2}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}h(M_k)\right)^2.$$

It is clear that $\varepsilon_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_W$. Since $\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}h(M_k)\overline{\psi}_i(M)$ is convergent series in the sense of $\|\cdot\|_W$, thus $\|\varepsilon_n\|_W \to 0$, $(n \to \infty)$. □

4. Numerical results

The purpose of this section is to present some examples to illustrate computationally the results established in the paper. The approximate solution $u_n(x, t)$ is calculated by (3.5).

Example 1. For an example problem from [14,1], take

$$\begin{cases}\lambda_0 = 1, \quad \gamma_0 = 0, \quad \lambda_1 = 1, \quad \gamma_1 = 0,\\ g(x,t) = \dfrac{-2(x^2 + t + 1)}{(t+1)^3}, \quad f(x) = x^2, \quad p_0(x) = x,\\ p_1(x) = x, \quad q_0(t) = -\dfrac{1}{4(t+1)^2}, \quad q_1(t) = \dfrac{3}{4(t+1)^2}\end{cases} \tag{4.1}$$

where $0 < x < 1$, $0 < t < 1$, the exact solution is $u(x, t) = (\frac{x}{t+1})^2$. In [1], The author presented two-level Crandall's implicit scheme, the three-level Dufort–Frankel explicit method, BTCS implicit method and FTCS method for solving Eq. (4.1). The domain $[0, 1] \times [0, T]$ is divided into an $M \times N$ mesh with the spatial step size $h = 1/M$ in x direction and the time step size $k = T/N$, respectively. Grid points (x_i, t_n) are defined by $x_i = ih, i = 0, 1, 2, \ldots, M, t_n = nk, n = 0, 1, 2, \ldots, N$, in which M and N are integers. $s = k/h^2$. The author present the relative error for $u(0.60, 1.0)$ in $[0, 1] \times [0, 1]$, with $h = 0.05, 0.01, 0.0025, 0.005, 0.001$, and $s = 0.40$, namely $k = 0.001, 0.00004, 0.0000025, 0.00001, 0.0000004$. Using our method, by (3.5), we get the approximate solution $u_n(x, t)(\Delta x = h = 0.05, 0.025\Delta t = k = 0.1, 0.05, 0.025, 0.02, n = 200, 400, 800, 1000, 1600, 2000)$ in $[0, 1] \times [0, 1]$. Comparing the relative errors of the numerical values of $u(0.60, 1.0)$ by our method with the relative errors of the numerical values of $u(0.60, 1.0)$ by [1] in Table 1. Using our method, the mean root square errors of $u(x, t)$ and CPU time are shown in Table 2. The mean root square errors of derivatives of $u(x, t)$ are shown in Table 3.

Example 2. We consider the following boundary value problem [1]

$$\begin{cases}\lambda_0 = 1, \quad \gamma_0 = 0, \quad \lambda_1 = 1, \quad \gamma_1 = 0,\\ g(x,t) = -\mathrm{e}^{-(x+\sin t)}(1 + \cos t), \quad f(x) = \mathrm{e}^{-x},\\ p_0(x) = 3.784423x, \quad p_1(x) = 0.6623722\cos(x), \quad q_0(t) = 0, \quad q_1(t) = 0\end{cases}$$

Table 1
Relative errors of the numerical values of $u(0.60, 1.0)$ in our method and in [1], for Example 1.

h	Our method ($k = 0.1$)	Our method in ($k = 0.05$)	Our method ($k = 0.025$)	Our method ($k = 0.02$)
0.05	2.4×10^{-2}	1.9×10^{-3}	2.3×10^{-4}	2.1×10^{-4}
0.025	2.7×10^{-2}	3.3×10^{-3}	2.3×10^{-4}	4.0×10^{-5}
h	BTCS in [1]	Crandall in [1]	FTCS in [1]	Dufort–Franke in [1]
0.05	7.3×10^{-2}	3.8×10^{-3}	7.5×10^{-2}	7.8×10^{-2}
0.025	1.8×10^{-2}	2.1×10^{-4}	1.9×10^{-2}	1.9×10^{-2}

Table 2
The mean root square errors of $u(x, t)$ and CPU time in our method for Example 1.

(h, k)	(0.05, 0.1)	(0.05, 0.05)	(0.05, 0.025)	(0.025, 0.1)	(0.025, 0.05)	(0.025, 0.025)
Mean root square errors	1.2×10^{-3}	2.3×10^{-4}	1.1×10^{-4}	1.3×10^{-3}	1.9×10^{-4}	3.9×10^{-5}
CPU time (s)	1.2	2.1	12.6	3.3	12.1	80.5

Table 3
The mean root square errors of derivatives of $u(x, t)$ in our method for Example 1.

(h, k)	(0.05, 0.1)	(0.05, 0.05)	(0.05, 0.025)	(0.025, 0.1)	(0.025, 0.05)	(0.025, 0.025)
$\left[\sum_{i=1}^{n}(\partial_t u - \partial_t u_n)^2/n\right]^{1/2}$	2.1×10^{-3}	2.9×10^{-4}	1.7×10^{-4}	2.2×10^{-3}	2.9×10^{-4}	5.4×10^{-5}
$\left[\sum_{i=1}^{n}(\partial_x u - \partial_x u_n)^2/n\right]^{1/2}$	4.0×10^{-3}	1.2×10^{-3}	4.3×10^{-4}	3.1×10^{-3}	9.1×10^{-4}	3.1×10^{-4}
$\left[\sum_{i=1}^{n}(\partial_{xx}^2 u - \partial_{xx}^2 u_n)^2/n\right]^{1/2}$	1.6×10^{-2}	4.5×10^{-3}	9.9×10^{-4}	1.1×10^{-2}	3.4×10^{-3}	8.5×10^{-4}
$\left[\sum_{i=1}^{n}(\partial_{xt}^2 u - \partial_{xt}^2 u_n)^2/n\right]^{1/2}$	6.6×10^{-3}	1.9×10^{-3}	6.4×10^{-4}	5.4×10^{-3}	1.4×10^{-3}	4.8×10^{-4}

Table 4
Relative errors of the numerical values of $u(0.60, 0.10)$ in our method and in [1], for Example 2.

h	Our method ($k = 0.1$)	Our method in ($k = 0.05$)	Our method ($k = 0.025$)	Our method ($k = 0.02$)
0.05	1.3×10^{-4}	7.2×10^{-5}	4.7×10^{-5}	2.7×10^{-5}
0.025	1.6×10^{-5}	2.6×10^{-5}	1.4×10^{-5}	3.2×10^{-5}
h	BTCS in [1]	Crandall in [1]	FTCS in [1]	Dufort–Franke in [1]
0.05	6.3×10^{-2}	3.9×10^{-3}	6.4×10^{-2}	6.8×10^{-2}
0.025	1.5×10^{-2}	2.4×10^{-4}	1.6×10^{-2}	1.7×10^{-2}

Table 5
The mean root square errors of $u(x, t)$ and CPU time in our method for Example 2.

(h, k)	(0.05, 0.1)	(0.05, 0.05)	(0.05, 0.025)	(0.025, 0.1)	(0.025, 0.05)	(0.025, 0.025)
Mean root square errors	2.3×10^{-3}	7.0×10^{-4}	1.0×10^{-3}	2.0×10^{-3}	4.2×10^{-4}	2.7×10^{-4}
CPU time (s)	1.1	2.2	12.3	3.1	12.1	80

Table 6
The mean root square errors of derivatives of $u(x, t)$ in our method for Example 2.

(h, k)	(0.05, 0.1)	(0.05, 0.05)	(0.05, 0.025)	(0.025, 0.1)	(0.025, 0.05)	(0.025, 0.025)
$\left[\sum_{i=1}^{n}(\partial_t u - \partial_t u_n)^2/n\right]^{1/2}$	6.9×10^{-3}	1.8×10^{-3}	3.3×10^{-3}	6.1×10^{-3}	1.2×10^{-3}	7.7×10^{-3}
$\left[\sum_{i=1}^{n}(\partial_x u - \partial_x u_n)^2/n\right]^{1/2}$	2.5×10^{-3}	7.3×10^{-4}	1.1×10^{-3}	2.1×10^{-3}	4.5×10^{-4}	3.0×10^{-4}
$\left[\sum_{i=1}^{n}(\partial_{xx}^2 u - \partial_{xx}^2 u_n)^2/n\right]^{1/2}$	5.1×10^{-3}	1.2×10^{-3}	2.8×10^{-4}	4.4×10^{-3}	9.8×10^{-4}	6.1×10^{-4}
$\left[\sum_{i=1}^{n}(\partial_{xt}^2 u - \partial_{xt}^2 u_n)^2/n\right]^{1/2}$	7.3×10^{-3}	1.9×10^{-3}	3.5×10^{-4}	6.4×10^{-3}	1.3×10^{-3}	8.1×10^{-4}

where $0 < x < 1, 0 < t < 1$. The exact solution is $u(x, t) = e^{-(x+\sin t)}$. In terms of (3.5), we calculate the approximate solution $u_n(x, t)(\Delta x = h = 0.05, 0.025 \Delta t = k = 0.1, 0.05, 0.025, 0.02, n = 200, 400, 800, 1000, 1600, 2000)$ in $[0, 1] \times [0, 1]$. Comparing the relative errors of the numerical values of $u(0.60, 0.10)$ by our method with the relative errors of the numerical values of $u(0.60, 0.10)$ by [1] in Table 4. Using our method, the mean root square errors of $u(x, t)$ and CPU time are shown in Table 5. The mean root square errors of derivatives of $u(x, t)$ are shown in Table 6.

From the above results, we see that our method uses bigger spatial length, few CPU time and obtain good results.

5. Summary and concluding remarks

In this paper, a new algorithm was applied to the non-classical parabolic problems arise in engineering applications. The algorithm, which may be implemented based on a reproducing kernel space. We construct the reproducing kernel space, in which the given initial conditions and nonlocal boundary conditions of the non-classical parabolic problems can be involved. This is a smart technique to deal with the non-classical conditions. It is worthy to note that, in our work, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. By constructing different reproducing kernel spaces to deal with the initial condition and nonlocal boundary conditions, the method used in the paper can be generalized to the other appropriate partial differential equations subject to the initial condition and nonlocal boundary conditions.

Appendix

A.1. The method for solving the reproducing kernel function $R_y(x)$ in $W_0[0, 1]$

By (2.1),

$$\begin{aligned}\langle u(x), R_y(x)\rangle_{W_1} &= \sum_{i=0}^{2} u^{(i)}(0)\partial_x^i R_y(0) + \int_0^1 u'''(x)\partial_x^3 R_y(x)\mathrm{d}x\\ &= \sum_{i=0}^{2} u^{(i)}(0)\partial_x^i R_y(0) + \int_0^1 u'''(x)\partial_x^3 R_y(x)\mathrm{d}x + c_1\int_0^1 p_0(x)u(x)\mathrm{d}x - c_1\lambda_0 u(0) - c_1\gamma_0 u'(0)\\ &\quad + c_2\int_0^1 p_1(x)u(x)\mathrm{d}x - c_2\lambda_1 u(1) - c_1\gamma_1 u'(1)\\ &= u(0)(R_y(0) - \partial_x^5 R_y(0) + c_1\lambda_0) + u'(0)(\partial_x R_y(0) + \partial_x^4 R_y(0) + c_1\gamma_0) + u''(0)(\partial_x^2 R_y(0) - \partial_x^3 R_y(0))\\ &\quad + u(1)(\partial_x^5 R_y(1) + c_2\lambda_1) - u'(1)(\partial_x^4 R_y(1) + c_2\gamma_1) + u''(1)\partial_x^3 R_y(1)\\ &\quad - \int_0^1 u(x)(\partial_x^6 R_y(x) - c_1 p_0(x) - c_2 p_1(x))\mathrm{d}x.\end{aligned}$$

Let

$$\partial_x^6 R_y(x) - c_1 p_0(x) - c_2 p_1(x) = -\delta(x-y), \tag{A.1}$$

$$\begin{cases}\partial_x^i R_y(0) - \partial_x^5 R_y(0) + c_1\lambda_0 = 0,\\ \partial_x R_y(0) + \partial_x^4 R_y(0) + c_1\gamma_0 = 0,\\ \partial_x^2 R_y(0) - \partial_x^3 R_y(0) = 0,\\ \partial_x^5 R_y(1) + c_2\lambda_1 = 0,\\ \partial_x^4 R_y(1) + c_2\gamma_1 = 0,\\ \partial_x^3 R_y(1) = 0,\end{cases} \tag{A.2}$$

then $\langle u(x), R_y(x)\rangle_{W_1} = u(y)$, $R_y(x)$ can be represented by

$$R_y(x) = \begin{cases}\sum_{i=0}^{5} a_i x^i + c_1 P_0(x) + c_2 P_1(x), & x \le y,\\ \sum_{i=0}^{5} b_i x^i + c_1 P_0(x) + c_2 P_1(x), & x > y,\end{cases} \tag{A.3}$$

where $P_0(x) = \underbrace{\int_0^x \cdots \int_0^x}_{6} p_0(x)\,\underbrace{\mathrm{d}x\cdots\mathrm{d}x}_{6}$, $P_1(x) = \underbrace{\int_0^x \cdots \int_0^x}_{6} p_1(x)\,\underbrace{\mathrm{d}x\cdots\mathrm{d}x}_{6}$. From Eq. (A.1) and the definition of $W_0[0, 1]$, we have

$$\begin{cases}\lim_{x\to y^+}\partial_x^i R_y(x) = \lim_{x\to y^-}\partial_x^i R_y(x), \quad i = 0, 1, \ldots, 4,\\ \lim_{x\to y^+}\partial_x^5 R_y(x) - \lim_{x\to y^-}\partial_x^5 R_y(x) = -1,\\ \int_0^1 p_0(x)u(x)\mathrm{d}x - \lambda_0 u(0) - \gamma_0 u'(0) = 0,\\ \int_0^1 p_1(x)u(x)\mathrm{d}x - \lambda_1 u(1) - \gamma_1 u'(1) = 0.\end{cases} \tag{A.4}$$

According to Eqs. (A.2) and (A.4) $a_i, b_i, (i = 0, \ldots, 5), c_1, c_2$ can be solved easily, and therefore the reproducing kernel function $R_y(x)$ is obtained.

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